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LETTER TO THE EDITOR

An exactly soluble moving-mirror problem

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Abstract. We present for the first time an exact treatment of the production of radiation by a mirror which accelerates along a smooth, asymptotically static trajectory in two-dimensional Minkowski space. The treatment confirms the general expectations of earlier work which was based on questionable approximations and unrealistic asymptotic behaviour.

In the recent spate of investigations of quantum field theory in curved space and non-inertial coordinate systems, a useful heuristic device has been the so-called moving mirror (Fulling and Davies 1976, Davies and Fulling 1977, see Birrell and Davies 1982 for a review). In its simplest form, this consists of a perfectly reflecting boundary (point) in two-dimensional Minkowski space (t, x) which moves along some non-trivial trajectory

$$x = z(t). \tag{1}$$

A massless scalar field ϕ satisfying the equation

$$\Box\phi = 0 \tag{2}$$

is constrained to vanish at the 'mirror'.

If the mirror is initially static in some inertial frame, one may define positive-frequency 'in' modes of the form

$$(\pi\omega)^{-1/2} \sin \omega x \, \mathrm{e}^{-\mathrm{i}\omega t} \tag{3}$$

and an associated vacuum state $|0_{\text{in}}\rangle$ in the usual way. The state $|0_{\text{in}}\rangle$ is the usual no-particle state of ordinary quantum field theory in the presence of a static reflecting boundary. A uniformly moving particle detector, for example, will register no particles for this state with unit probability.

Once the mirror starts to move about, however, the vacuum is disturbed and particles are created. To the right of the moving mirror the flow of energy is given by

$$F(u) = (24\pi)^{-1}[p'''p' - \tfrac{3}{2}(p'')^2]/(p')^2 \tag{4}$$

where

$$p(u) = 2\tau_u - u, \qquad \tau_u - z(\tau_u) = u. \tag{5}$$

The flux is a function of the retarded time $u \equiv t - x$ only because the initial conditions rule out any incoming radiation.

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The particle flux is computed from the Bogolubov transformation between modes (3) and the complicated outgoing modes which have the form

$$(4\pi\omega)^{-1/2}(e^{-i\omega v} - e^{-i\omega p(u)}) \tag{6}$$

where $v \equiv t + x$.

In practice it is very hard to find trajectories $z(t)$ for which this Bogolubov transformation may be evaluated explicitly. Two cases of interest, however, are the asymptotically null trajectories

$$z(t) = -t - A\, e^{-2\kappa t} + B, \qquad \text{as } t \to \infty, \tag{7}$$

$$z(t) = c - (c^2 + t^2)^{1/2}, \qquad t > 0, \tag{8}$$

where A, B, c and κ are positive constants. Both trajectories have to be joined on smoothly to the static world line of the mirror in the region $t \to -\infty$.

Case (7) yields a Planck spectrum (thermal radiation) with a temperature $\kappa/2\pi$, but because (7) is valid only for late times, obscure approximations are needed. Case (8) yields a Bessel function type spectrum, which is of considerable interest because the flux F vanishes for this trajectory. This example therefore illustrates a peculiar relationship between 'particles' and energy. However, the need to join smoothly onto the static trajectory in the past involves a departure from the form of (8). Alternatively the function (8) may be joined in a C^1 way to $z = 0$ at $t < 0$, but there is then a δ-function pulse in the flux from the mirror at $t = 0$. Add to these difficulties the fact that the asymptotically null trajectories are non-physical, and also bring a crop of problems to do with completeness of the field modes on $\mathscr{I}^+$ (see Davies and Fulling 1977), and it is clear how desirable it is to study an exactly soluble model that does not involve these dubious features.

We have found such a model. Consider the trajectory

$$t = -z \pm A(e^{-2z/B} - 1)^{1/2} \tag{9}$$

where A and B are constants, and $A > B$ (required so that $|\dot{z}| < 1$). The shape of this trajectory is shown in figure 1. The + sign in (9) refers to the upper half of the curve, the − sign to the lower half (it is not symmetric about $t = 0$). The curve is C^∞ everywhere, and so we avoid the 'joining' problems of the previous examples. It is also asymptotically static: as $t \to \pm\infty$, the mirror velocity approaches zero at $z \to -\infty$. The mirror therefore approaches at accelerating speed from the far left, decelerates, and then recedes to infinity, with $z \leqslant 0$.

The energy flux is found from (4) and (5) to be

$$\langle T^{01}(u)\rangle = (B/6\pi)(x^5 + \tfrac{1}{2}Bx^4 - 2A^2x^3 - 3BA^2x^3 - 3A^4x - \tfrac{3}{2}A^4B)/(x^2 + 2Bx + A^2)^4, \tag{10}$$

where

$$u = B\ln(x^2/A^2 + 1) + x. \tag{11}$$

This function is shown in figure 2.

The integrated energy emitted during the whole motion is

$$\int_{z(t)}^{\infty} \langle T^{00}(t, x)\rangle\, dx = \frac{B^2}{48(A^2 - B^2)^{3/2}}. \tag{12}$$

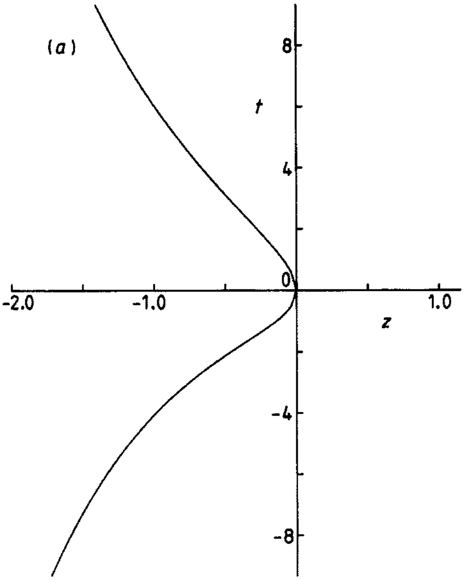

Figure 1. (a) The mirror trajectory for $A=2$, $B=1$. (b) The mirror trajectory for $A=4$, $B=1$.

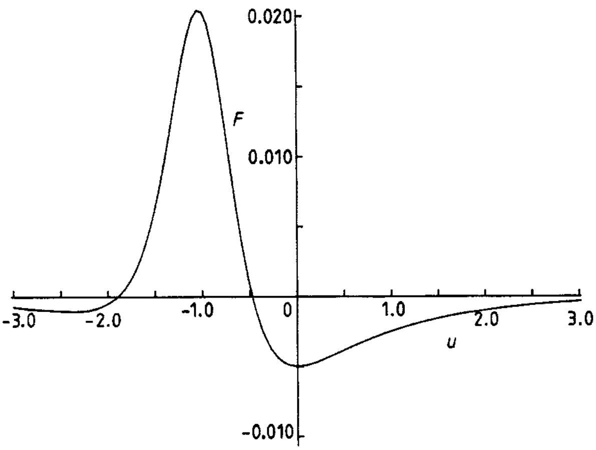

Figure 2. The energy flux, F, from the mirror ($A=2$, $B=1$).

Notice that as $A \to B$ the integrated energy $\to \infty$. This corresponds to the trajectory becoming null at one point. Also as $B \to 0$ the trajectory 'straightens out' and the integrated energy $\to 0$.

The Bogolubov transformation reduces to the integrals (Davies and Fulling 1977)

$$\left.\begin{matrix}\alpha_{\omega'\omega}\\ \beta_{\omega'\omega}\end{matrix}\right\}=\frac{1}{2\pi}\left(\frac{\omega'}{\omega}\right)^{1/2}\int_{-\infty}^{\infty}\exp[-\mathrm{i}\omega' x \pm \mathrm{i}\omega f(x)]\,\mathrm{d}x \tag{13}$$

where $f(x)=u$, given by (11), and is the inverse of the function $p(u)$ given in (5).

The integral may be performed in terms of modified Bessel functions (Gradshteyn and Ryzhik 1965) to yield for the Bogolubov coefficients

$$\left.\begin{matrix}\alpha_{\omega'\omega}\\ \beta_{\omega'\omega}\end{matrix}\right\} = \pm \mathrm{i}\pi^{-3/2}\left(\frac{\omega'}{\omega}\right)^{1/2}\left(\frac{2}{\omega'\mp\omega}\right)^{1/2\pm\mathrm{i}\omega B}$$

$$\times A^{1/2\mp\mathrm{i}\omega B}\sinh(\pi\omega B)\Gamma(1\pm\mathrm{i}\omega B)K_{-1/2\mp\mathrm{i}\omega B}[A(\omega'\mp\omega)] \tag{14}$$

whence

$$|\beta_{\omega'\omega}|^2 = \frac{2AB}{\pi^2}\left(\frac{\omega'}{\omega'+\omega}\right)\sinh(\pi\omega B)|K_{-1/2+\mathrm{i}\omega B}[A(\omega'+\omega)]|^2. \tag{15}$$

The number of particles created in mode ω is given by

$$n(\omega) = \int_0^\infty |\beta_{\omega'\omega}|^2\,\mathrm{d}\omega'. \tag{16}$$

We have not been able to evaluate (16) explicitly, but it is easy to verify that it is finite. One expects on general grounds that (15) should decline exponentially in both ω and ω' as these quantities $\rightarrow\infty$. This is so, as may be deduced from the asymptotic properties of the K function.

These results are shown in figure 3 which is a computer plot of (15) as a function of ω for fixed ω', and also as a function of ω' for fixed ω.

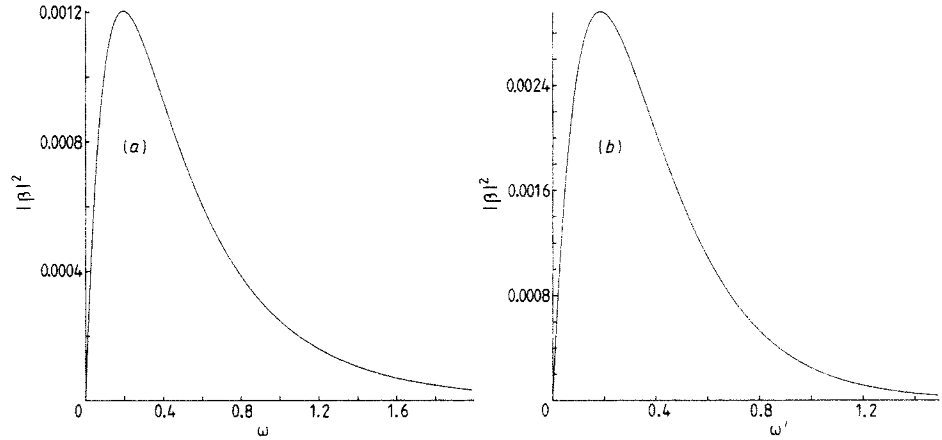

Figure 3. (a) $|\beta|^2$ as a function of ω for fixed $\omega' = 1.0$ ($A = 2$, $B = 1$). (b) $|\beta|^2$ as a function of ω' for fixed $\omega = 1.0$ ($A = 2$, $B = 1$).

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